The Stability of Abstract Boundary Essential Singularities

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Abstract

The abstract boundary has, in recent years, proved a general and flexible way to define the singularities of space-time. In this approach an essential singularity is a non-regular boundary point of an embedding which is accessible by a chosen family of curves within finite parameter distance. Ashley and Scott proved the first theorem relating essential singularities in strongly causal space-times to causal geodesic incompleteness. Linking this with the work of Beem on the C^r-stability of geodesic incompleteness allows proof of the stability of these singularities. Here I present this result stating the conditions under which essential singularities are C^1-stable against perturbations of the metric.

Keywords: abstract boundary, essential singularity, stability, space-time

1 Introduction

The stability of the physical features of space-time has been a significant area of inquiry since the production of the first exact solutions of the Einstein equation. It has always been thought that for a given space-time to be physically reasonable, or for a given feature to exist in the universe at large, that the space-time be robust against perturbations of the metric. The issue of stability has, however, been somewhat difficult to define in the abstract sense of a pseudo-Riemannian manifold since there has never been a completely coordinate invariant method of defining metrics that are *near* one another. In the practical mathematical and geometrical sense the difficulty has arisen because at present there is no candidate for a topology on the space of metrics over a given manifold preserving coordinate invariance, which is a crucial feature of general relativity. Nevertheless with strict conditions on the allowed coverings of coordinate charts for a space-time it is possible to derive some important results. With this in mind I will use the Whitney C^r-fine topology on the space of metrics and restate the notion of C^r-stability for some feature of a space-time. I will then proceed to present some physically intuitive examples of the use of the C^r-fine topology with a view to summarising the literature on the stability of geodesic completeness/incompleteness relevant to creating a stability theorem for abstract singularities. Finally I will quote the result of Ashley and Scott and present the stability theorem for abstract boundary singularities.

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2 A review of the Whitney C^r-fine topology on the space of metrics

Let $\mathcal{A} = \{\mathcal{U}_i\}$ be a chosen fixed covering of $\mathcal{M}$ by a countable collection of charts of $\mathcal{M}$. We will also assume that every chart has compact closure in a larger chart, (i.e. for all i, $\overline{\mathcal{U}_i}$ is compact, and there exists a $\mathcal{V}_j$ so that $\overline{\mathcal{U}_i} \subset \mathcal{V}_j$, with $\{\mathcal{V}_j\}$ forming an atlas on $\mathcal{M}$) and that the covering of $\mathcal{M}$ by the $\mathcal{U}_i$ is locally finite. Now let $\epsilon : \mathcal{M} \to \mathbb{R}^+$ be a continuous function.

Definition 1
For any two Lorentzian metrics g, h, we write,

$$|g - h|_{r,\mathcal{P}} < \epsilon, \tag{1}$$

if for each point $p \in \mathcal{P} \subset \mathcal{M}$,

$$|g_{ab} - h_{ab}| < \epsilon(p) \text{ and } |g_{ab,c_1 c_2 \ldots c_r} - h_{ab,c_1 c_2 \ldots c_r}| < \epsilon(p) \tag{2}$$

when the metric is evaluated at p for all indices a, b, c_1, c_2, $\ldots$, c_r in all the given charts $\mathcal{U}_i \in \mathcal{A}$ which contain p.

Definition 2 (Whitney C^r-fine topologies)
The *Whitney C^r-fine topologies* (or simply the *C^r-fine topologies*) are defined by basis neighbourhoods of the form

$$\mathcal{N}(g, \epsilon) := \{h : |g - h|_{r,\mathcal{M}} < \epsilon\} \tag{3}$$

about each metric g and for each continuous function $\epsilon : \mathcal{M} \to \mathbb{R}^+$.

If $h \in \mathcal{N}(g, \epsilon)$ for some given metric g, then the two metrics g and h are termed *C^r-close.*

Definition 3 (C^r-stable property)
A property, X, of a space-time, $(\mathcal{M}, g)$, is termed C^r-stable if it is true for every metric in some open neighbourhood of g in the C^r-fine topology.

The C^r-fine topology may be shown to be independent of the cover $\{\mathcal{U}_i\}$ if the conditions above are satisfied. However, it is worthwhile noting how these conditions are used to restrict the allowed coverings. For example, if the covering is not locally finite then it is possible that there would be no partition of unity available over the charts to guarantee consistent definition of the metric. It follows that ϵ-neighbourhoods of that metric would be ill-defined. Similarly if one could not guarantee that a chart had compact closure in a chart of another atlas then ϵ may not possess a maximum. This result would then make it impossible to guarantee that the metrics and/or their derivatives would have their deviation confined.

It is also worthwhile noting that coverings of this sort exist for all except the most pathological of examples and so these conditions do not really pose much of an impediment to the practical use of the C^r-fine topology. Consequently, at present, the C^r-fine topology over metrics is arguably the most straightforward and practical notion for the *nearness* of metrics.

3 A physical interpretation of the C^r-fine topologies

I will now digress to give an intuitive description of the C^r-fine topology. For the cases $r = 0, 1, 2$ it is relatively simple to visualise the physical relationship between C^r-close metrics.

Example 1 If two metrics g, h for the space-times $(\mathcal{M}, g)$ and $(\mathcal{M}, h)$ are C^0-close, then their metric components are close, implying that the light cones of equivalent points in both space-times are *close*.

Example 2 If two metrics g, h for the space-times $(\mathcal{M}, g)$ and $(\mathcal{M}, h)$ are C^1-close, then the metric components and their first derivatives (and hence the Christoffel symbol functions on $\mathcal{M}$, $\Gamma^a{}_{bc}(x)$) are close. This implies, by the continuous depedence of the solutions of the geodesic equation on the Christoffel symbols, that the geodesic systems under both metrics are *close* in addition to the light cone structure. One should consult Beem, Ehrlich and Easley [3, p. 247] for additional references.

Example 3 If two metrics g, h are C^2-close, then additionally we have that the second derivatives of the metric are close for both metrics and hence the components of the Riemann curvature tensor and other Riemann derived objects (e.g. curvature invariants, Ricci, R_{ab}, and Weyl tensors, C_{abcd}) are *close*.

One should now also be able to extrapolate the above examples to higher derivatives. For example, if two metrics are C^3-close then we would expect that in addition to the light cones, geodesic systems and curvature being close that the first derivatives of the Riemann tensor would also be close. By analogy we can produce interpretations for C^r-fine topologies with larger r. Note that as r increases, more metrics are excluded from any given ϵ-neighbourhood, $\mathcal{N}(g, \epsilon)$, and the resulting topologies about the given metric are *finer*. Correspondingly any property proved to be C^s-stable will also be C^r-stable, whenever $s \leqslant r < \infty$. It is also important to remember that the C^r-fine topologies are in a certain sense too coarse since they also include too many metrics which one may not wish to consider close[1]. For example, one may choose ϵ functions which are very small in some compact region of $\mathcal{M}$ but are far from zero elsewhere. The resulting C^r-fine open neighbourhood, $\mathcal{N}(g, \epsilon)$, will contain not only metrics whose values and derivatives are close to those of g everywhere in $\mathcal{M}$ but also those that deviate wildly outside of the compact region. An example of this behaviour is presented in Figure 1.

An important application of the C^r-fine topology can be seen in the analysis of the causal structure of space-times. Using the above described notion that if two Lorentzian metrics are C^0-close, then their light cones are close, we obtain the following precise definition of the stable causality property for space-time (as motivated by Geroch [1, p. 241]):

Definition 4 (stable causality)
A space-time $(\mathcal{M}, g)$ is *stably causal* if there exists a C^0-fine neighbourhood, $U(g)$, of the Lorentzian metric, g, such that for each $h \in U(g)$, $(\mathcal{M}, h)$ is causal.

[1] The metrics allowed in these C^r-fine neighbourhoods may correspond to non-physical curvature sources. A more detailed discussion follows in §5.

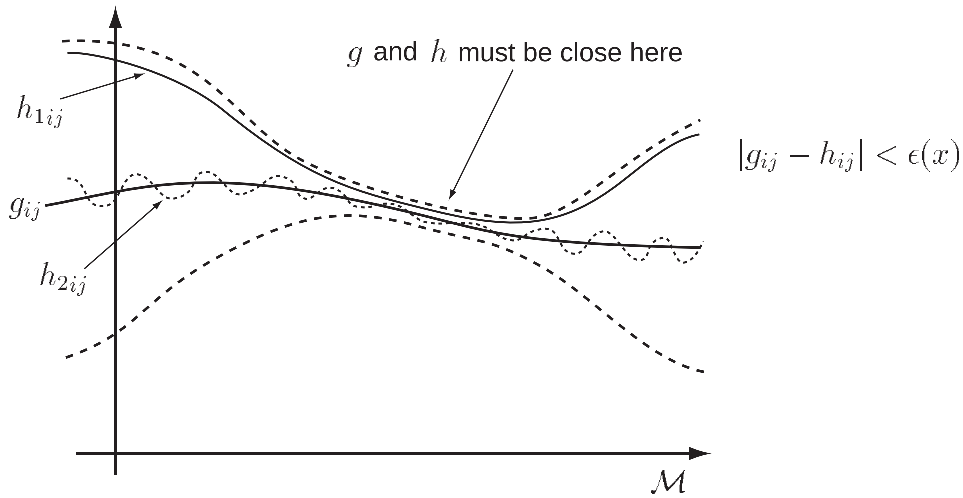

Figure 1: The figure compares the values of a metric component from g and two C^0-close metrics, h_1 and h_2. Note that the metric component, h_{1ij}, can vary significantly from g_{ij} since ϵ is only small in a compact region. The metric component, h_{2ij}, has values ϵ-close to g_{ij} but has a wildly varying first derivative. Hence the topology allows both h_{1ij} and h_{2ij} to be close to g_{ij} even though it appears that only h_{2ij} should be included. Consequently more metrics are present in $\mathcal{N}(g, \epsilon)$ than one's intuition might indicate. One may also want for h_{2ij} to be excluded from this ϵ-neighbourhood due to its wildly deviating derivative. This metric could be eliminated by choosing the topology to be C^1-fine, since then the neighbourhood would not contain metrics where the slope differed more than ϵ from that of g_{ij}. One can, however, easily devise examples where the second and higher derivatives behave pathologically. Seeking greater values of r for the C^r-fine topologies would lead to even finer topologies and exclude these cases.

Hence stably causal space-times remain causal under small C^0-fine perturbations of the metric. The reader should note that Definition 4 (from Beem, Ehrlich and Easley [3, p. 63]) redefines Geroch's idea of 'the spreading of light cones' precisely and the interested reader is asked to compare this with the alternate definition given in Hawking and Ellis [4, p. 198].

3.1 The stability of geodesic completeness/incompleteness

The stability of geodesic completeness/incompleteness has, over the years, been investigated closely by Beem and Ehrlich [5] and also Williams [6]. A thorough review of the literature on the stability of completeness and incompleteness is provided in Beem, Ehrlich and Easley [3, p. 239-270].

Examples by Williams [6] show that both geodesic completeness and geodesic incompleteness are not C^r-stable for space-time, in general. In addition, Williams also provided examples showing that these properties may fail to be stable even for compact/non-compact space-times. Of course this still leaves the possibility that with additional constraints made on the space-time that geodesic completeness/incompleteness may be stable. Beem showed that geodesic incompleteness is, in fact, C^1-stable for strongly causal space-times. This work is relevant to our task and the following presentation is designed to guide the reader unfamiliar with this result.

I will need precise notions of imprisonment and partial imprisonment for curves in

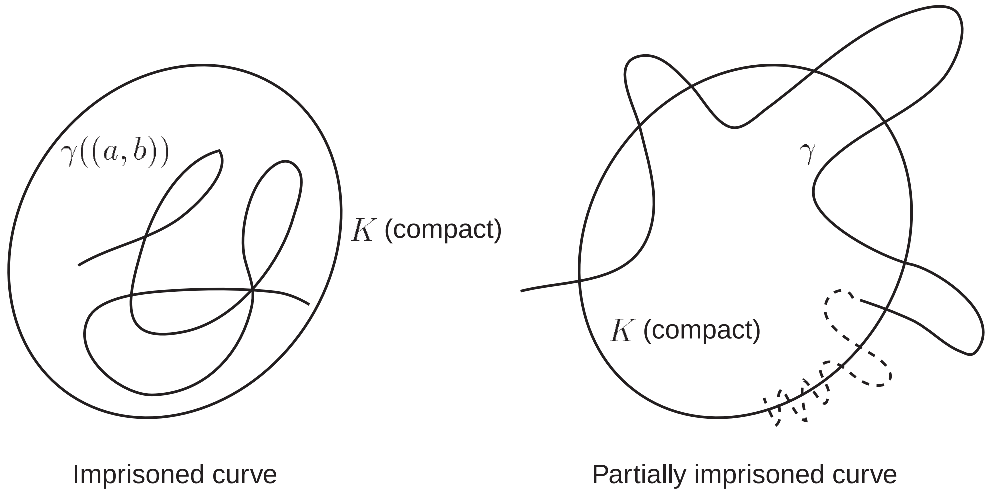

Figure 2: Imprisonment requires that the entire image of the curve be contained in some compact set whereas partial imprisonment requires that the curve exit and reenter a compact set an infinite number of times.

space-time since partial imprisonment is very closely related to the strong causality condition on a space-time. The following definitions are those used by Beem in the proof of the stability theorems.

Definition 5 (Imprisonment and Partial Imprisonment)
Let $\gamma : (a, b) \to \mathcal{M}$ be an inextendible geodesic.

1. The geodesic, γ, is *partially imprisoned* as $t \longrightarrow b$ if there is a compact set $K \subseteq \mathcal{M}$ and a sequence $\{x_i\}$ with $x_i \longrightarrow b$ from below such that $\gamma(x_i) \in K$ for all i.
2. The geodesic, γ, is *imprisoned* if there is a compact set K such that the entire image, $\gamma((a, b))$, is contained in K.

Essentially the definitions differ in that imprisonment implies that there is some compact set which encloses the entire curve $\gamma((a, b))$ while partial imprisonment only requires that there be an infinite subsequence of points which remains in the compact set. Hence for a partially imprisoned curve which is not totally imprisoned, the curve must not only continually reenter the compact set K, but must also exit it an infinite number of times (see Figure 2). Of course, a curve which is imprisoned is also partially imprisoned.

I now present the result of Beem [7] (see also Beem [3] p.265).

Theorem 1 Let $(\mathcal{M}, g)$ be a semi-Riemannian manifold. Assume that $(\mathcal{M}, g)$ has an endless geodesic $\gamma : (a, b) \to \mathcal{M}$ such that γ is incomplete in the forward direction (i.e. $b \neq \infty$). If γ is not partially imprisoned in any compact set as $t \longrightarrow b$, then there is a C^1-neighbourhood $\mathcal{U}(g)$ of g such that each g_1 in $\mathcal{U}(g)$ has at least one incomplete geodesic c. Furthermore, if γ is timelike (respectively, null, spacelike) then c may also be taken as timelike (respectively, null, spacelike).

Since strongly causal space-times do not allow past or future-directed non-spacelike curves to be partially imprisoned in any neighbourhood of a regular space-time point, Beem straightforwardly obtained the following corollary.

Corollary 1 If $(\mathcal{M}, g)$ is a strongly causal space-time which is causally geodesically incomplete, then there is a C^1-neighbourhood, $\mathcal{U}(g)$ of g, such that each g_1 in $\mathcal{U}(g)$ is causally geodesically incomplete.

4 A brief introduction to a-boundary essential singularities

The abstract boundary (or simply a-boundary) construction is a relatively recent addition to the collection of boundary constructions that have been applied to space-time. It provides a flexible structure for the classification of boundary points of embeddings and appears to bypass many of the problems common to the g, c and b-boundary constructions. Only those definitions necessary for understanding the stability result are included here. It is suggested that the reader consult Scott and Szekeres [8] for a more complete and comprehensive introduction to the a-boundary construction.

In the a-boundary picture the boundary points in question are the topological boundary points, $\partial_\phi \mathcal{M} := \partial(\phi(\mathcal{M}))$, of an open manifold, $\mathcal{M}$, under the image of a C^∞ embedding $\phi : \mathcal{M} \to \widehat{\mathcal{M}}$. It is important to note that both $\mathcal{M}$ and $\widehat{\mathcal{M}}$ are of the same dimension. Hence $\phi(\mathcal{M})$ is an open submanifold of $\widehat{\mathcal{M}}$. The ordered triple $(\mathcal{M}, \widehat{\mathcal{M}}, \phi)$ will from now on be termed an *envelopment*. Boundary points of different envelopments of the same manifold can turn up in different guises and it is useful to know when they are considered equivalent. For the following we will consider two boundary sets, $B \subset \partial_\phi \mathcal{M} \subset \widehat{\mathcal{M}}$ and $B' \subset \partial_\psi \mathcal{M} \subset \widehat{\mathcal{M}}'$ from two different envelopments. We say that B *covers* B', written $B \rhd B'$ if for every neighbourhood $\mathcal{U}(B) \subset \widehat{\mathcal{M}}$ [2] there exists a neighbourhood $\mathcal{U}'(B') \subset \widehat{\mathcal{M}}'$ such that $\phi \circ \psi^{-1}(\mathcal{U}' \cap \psi(\mathcal{M})) \subset \mathcal{U}$. This definition sums up the fact that any sequence approaching B' from within $\psi(\mathcal{M}) \subset \widehat{\mathcal{M}}'$ must have its image sequence (i.e. mapped through $\phi \circ \psi^{-1}$) approach B. The covering relation obeys the conditions for a weak partial order and this leads us to the definition of equivalent boundary points, namely, $p \sim q$ iff $p \rhd q$ and $q \rhd p$.

The abstract boundary, $\mathcal{B}(\mathcal{M})$, is composed of equivalence classes (abstract boundary points) of boundary sets equivalent to a boundary point in some envelopment. One should note that this basic structure is independent of the existence of a metric or chosen family of curves for the manifold and comes *gratis*. In order to classify abstract boundary points further we will need to choose a family of curves, $\mathcal{C}$, obeying the *bounded parameter property*. The technical details of the importance of the bounded parameter property can be found in Scott and Szekeres [8]. However it suffices to say that the curves we will choose, namely, the family of affinely parametrised causal geodesics do satisfy this condition. If there is a representative of the a-boundary point equivalence class which is the limit point[3] of some curve in the family, then the a-boundary point is termed *$\mathcal{C}$-approachable*. This definition is internally consistent due to the formulation of the covering relation (see Theorem 17 of Scott and Szekeres [8]).

If we provide the manifold with the additional structure of a metric then we can continue to classify the boundary points of an embedding by asking whether there exists an extension of the metric about the boundary point in the new envelopment. Consequently we now assume the manifold to be endowed with a pseudo-Riemannian metric, g. An *extension* of a manifold is defined as an envelopment of a pseudo-Riemannian manifold, $(\mathcal{M}, g)$, by a second pseudo-Riemannian manifold, $(\widehat{\mathcal{M}}, \hat{g})$, with

[2] Note that $\mathcal{U}(B)$ means that $\mathcal{U}$ is a neighbourhood of B.

[3] Other authors may term this an *accumulation point* or *cluster point*.

embedding ϕ such that $\hat{g}|_{\phi(\mathcal{M})} = (\phi^{-1})^* g$. The extension will be denoted by the ordered quintuple $(\mathcal{M}, g, \widehat{\mathcal{M}}, \hat{g}, \phi)$. This definition simply requires that the metric over $\widehat{\mathcal{M}}$ agrees with the induced metric from $\mathcal{M}$ on $\phi(\mathcal{M})$. One should also note that this definition is consistent (but not equivalent) with the notion of metric extension used in Hawking and Ellis [4]. Using this notion of metric extension we define a boundary point as being *regular for* g if there exists a pseudo-Riemannian manifold $(\overline{\mathcal{M}}, \bar{g})$ such that $\phi(\mathcal{M}) \cup \{p\} \subseteq \overline{\mathcal{M}} \subseteq \widehat{\mathcal{M}}$ and $(\mathcal{M}, g, \overline{\mathcal{M}}, \bar{g}, \phi)$ is an extension of $(\mathcal{M}, g)$. It is important to note that unlike the notion of $\mathcal{C}$-approachability, the regularity of an a-boundary point representative does not pass to the entire equivalence class since it is possible to choose poor envelopments in which representative boundary points are non-regular.

With these definitions in place we can now define singular boundary points.

Definition 6 (singular boundary point)
A boundary point $p \in \partial_\phi \mathcal{M}$ will be termed a *singular boundary point* or a *singularity* if

1. p is not a regular boundary point,
2. p is a $\mathcal{C}$-approachable point, and
3. there exists a curve, $\gamma \in \mathcal{C}$ which approaches p with bounded parameter.

Note that the definition of an a-boundary singularity is contingent on the choice of a suitable curve family, $\mathcal{C}$. Indeed, dependent on the choice of family, an abstract boundary point may be singular with respect to one family and non-singular with respect to another more restrictive class. For example, if we choose $\mathcal{C}$ to be the family of all general affinely parametrised causal curves, then we would define as singular all those non-regular points obeying the remaining conditions in Definition 6. However, if we were to use the family of affinely parametrised causal geodesics instead, then some of those points previously defined as singular could now be considered non-singular. These previously singular points, which now become non-singular, are those which are approachable by causal curves, but are unapproachable by causal geodesics. We will term a boundary set, B, *non-singular* if none of its points are singular. We are finally in a position now to define what is meant by an a-boundary essential singularity.

Definition 7 (essential singularity)
A singular boundary point p will be termed an *essential singularity* if it cannot be covered by a non-singular boundary set, B, of another embedding.

It is significant that despite the definition involving the concept of regularity, the property of being an essential singularity does pass through to all point members of an a-boundary equivalence class. Details of this are again found in Scott and Szekeres [8].

In summary, the essential singularities we will be considering are (i) non-regular boundary points of an embedding, which are (ii) limit points of some affinely parametrised causal geodesic reached in finite parameter distance and which (iii) cannot be removed by the existence of a second embedding having non-singular boundary points covering it. Physically this gives us the most fundamental idea of a real singularity as being the idealisation of a problem point of a space-time which is not a removable artifact and is 'tunable'[4] to the incompleteness of those curves, $\mathcal{C}$, considered physically significant.

[4]Although it is not explicitly stated above, the a-boundary can also be tuned to the level of differentiability

5 The relationship between abstract boundary singularities and causal geodesic incompleteness

With the definitions of the former section in hand, we now present the result of Ashley and Scott [2].

Theorem 2 Let $(\mathcal{M}, g)$ be a strongly causal, C^l maximally extended, C^k space-time $(1 \leqslant l \leqslant k)$ and $\mathcal{C}$ be the family of affinely parametrised causal geodesics in $(\mathcal{M}, g)$. Then $\mathcal{B}(\mathcal{M})$ contains a C^l essential singularity iff there is an incomplete causal geodesic in $(\mathcal{M}, g)$.

When considering the above theorem one must remember that it uses the technical definition of strong causality as defined in that paper. This definition is consistent with that used by Beem in the proof of Corollary 1 and with the notions of strong causality presented by Hawking and Ellis [4] and Penrose [9] (see Ashley and Scott [2]).

If we combine Corollary 1 with Theorem 2, while taking into account the degree of differentiability of the metric so that the C^1-fine topology is well-defined, then we find the following stability result for the presence of a-boundary essential singularities.

Theorem 3 (stability of abstract boundary essential singularities) Suppose there exists a C^k-essential singularity in $\mathcal{B}(\mathcal{M})$ for a C^k maximally extended, strongly causal space-time, $(\mathcal{M}, g)$ (where $1 \leqslant k$), with family $\mathcal{C}$ of affinely parametrised causal geodesics. Then there exists a C^1-fine neighbourhood, $\mathcal{U}(g)$ of g, so that for each g_1 in $\mathcal{U}(g)$, $\mathcal{B}(\mathcal{M})$ has a C^k-essential singularity for $(\mathcal{M}, g_1)$ provided $(\mathcal{M}, g_1)$ is also strongly causal and C^k-maximally extended for each g_1 in $\mathcal{U}(g)$.

The C^r-fine topologies defined earlier put bounds on the *geometrical* perturbations of the metric. Since they make no reference to the Einstein equation or stress-energy tensor, the metrics included in C^r-fine neighbourhoods will also include ones which are non-physical. These could include, for example, geometries whose equivalent matter source terms violate the strong energy condition. This is significant since energy conditions of this sort are used to prove the existence of incomplete timelike or null geodesics and consequently the existence of a-boundary essential singularities via Theorem 2. It is also possible to produce perturbations which do not coincide with the source of curvature for the original space-time. For example, if a geometric perturbation is made around a vacuum space-time then there is no guarantee that these variations will all possess a vacuum source.

Minkowski space is a useful example to consider the C^r-stability of the inextendability of a space-time. One perturbation that could be applied to this metric is the presence of small gravitational waves. It seems unlikely for gravitational radiation of a small amplitude that the causal structure and maximally extended nature of Minkowski space would be affected. On the other hand one could consider a Schwarzschild space-time which is perturbed by sending a small charge into the event horizon. One would expect to obtain a Reissner-Nördstrom space-time in this manner. Such a case seems very physical, however, we would obtain a space-time that originally was maximally extended but would change its global structure dramatically and hence, in a physical sense, the maximally extended nature of the Schwarzschild space-time cannot be con-

of the metric and of its extensions, as required. This will be significant later but not essential to the proof of the stability result.

sidered stable against these types of perturbation. In the case where a neighbourhood of maximally extended metrics does not exist we can imagine that a perturbation of the space-time metric may lead to the production of sets of extension hypersurfaces for the space-time.

It remains an open question to show that if a space-time is strongly causal then that property is C^1-stable. To the author's knowledge, this has not been proven in the literature. Intuitively, one might expect that strong causality should be C^r-stable, for some r, since strongly causal space-times do not allow the existence of causal curves which leave and return to a small neighbourhood of a manifold point. Thus it would be expected that there exist ϵ-neighbourhoods, whose perturbations allow causal curves passing near their own path, to probe out the exterior of the strong causality neighbourhoods of a manifold point.

6 Concluding remarks

At present, the abstract boundary construction proves to be the most promising construction with which to yield results about singularities in general relativity. The above stability result guarantees the stability of the existence of abstract boundary essential singularities provided we also have the stability of the strong causality and inextendability of the space-time in question. Consequently this theorem ensures the physicality of an essential singularity once one knows of the circumstances of its existence. Moreover, its proof is very straightforwardly obtained. It is hoped that the continuing stream of results involving the abstract boundary will bring it to the attention of Lorentzian geometers as a useful tool to apply to any question involving boundary points of space-time.

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References

[1] Geroch, R. and Horowitz, G. T. (Global structure of space-time). General Relativity: An Einstein Centenary Survey, in: S. Hawking and W. Israel, eds. Cambridge: Cambridge University Press, 1979, 212.

[2] Ashley, M. J. S. L. and Scott, S. M., *in preparation.*

[3] Beem, J. K., Ehrlich, P. E. and Easley, K. L. (1996). *Global Lorentzian Geometry*, Marcel Dekker Inc., New York.

[4] Hawking, S. W. and Ellis, G. F. R. (1973). *The Large Scale Structure of Space-time*, Cambridge University Press, Cambridge.

[5] Beem, J. K. and Ehrlich, P. E. (1987). Geodesic completeness and stability, *Math. Proc. Camb. Phil. Soc.*, **102**, 319-328.

[6] Williams, P. M. (1984). *Completeness and its stability on manifolds with connection*, Ph.D. dissertation, University of Lancaster.

[7] Beem, J. K. (1994). Stability of geodesic incompleteness, in: Differential Geometry and Mathematical Physics, Contemporary Math. Series, **170**, eds. J. K. Beem and K. L. Duggal, American Mathematical Society, Providence, RI.

[8] Scott, S. M. and Szekeres, P. (1994). The abstract boundary–a new approach to singularities of manifolds, *J. Geom. Phys.*, **13**, 223-253.

[9] Penrose, R. (1972). *Techniques of Differential Topology in Relativity*, Society for Industrial and Applied Mathematics, Philadelphia.